Composite Particle Theory in Quantum Electrodynamics

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Without use of pathintegral formalism a composite particle effective dynamics is developed for spinor quantum electrodynamics. By algebraic evaluation of spinor quantum electrodynamics in Coulomb gauge a corresponding functional equation is derived. The commutation rules for the transversal electromagnetic field can be deduced as a consequence of this formalism. By application of weak mapping theorems the QED functional equation can be mapped onto a functional equation for composite particles with mutual interaction and interaction with the electromagnetic field. The formalism is demonstrated for positronium states. The incorporation of renormalization into this scheme is verified.

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1. Introduction

Atoms and molecules in quantum electrodynamics, and nuclei and mesons in quantum chromodynamics are considered as composite particles built up of elementary leptons, baryons or quarks. Hence the study of the formation of composite particles and the derivation of their corresponding effective dynamics have to be some of the basic aims of abelian and nonabelian gauge theories with respect to their application to atomic and nuclear physics. In the course of the evaluation of these theories numerous investigations were made about the formation of bound states by means of Bethe-Salpeter equations and Schrödinger equations [1], and the effective dynamics were studied by Fock space methods [2] or path integral evaluation [3]. Summarizing the results of these investigations one observes the following drawbacks:

i) the bound state calculations are in general not related to a corresponding effective dynamics;
ii) the Fock space methods are in contradiction to algebraic representation theory of quantum fields [4];
iii) path integrals imply conceptional as well as technical difficulties, in particular with respect to effective dynamics for composite particles [5].

To avoid these drawbacks, in previous papers a composite particle theory was developed which is strictly based on the algebraic properties of quantum fields. One of the main ingredients of this theory is the weak mapping method which can be rigorously formulated by weak mapping theorems [6]. This method uses generating state functionals which can be defined by means of the field algebra. For every algebraic state on the field algebra there exists at least one regular cyclic representation in Hilbert space. The basis vectors of this representation are obtained by the application of symmetrized or antisymmetrized monomials of field operators on this cyclic Hilbert space vector (G.N.S.-construction). Generating state functionals are then defined as the set of projections of an arbitrary but fixed energy eigenstate in Hilbert space on these G.N.S.-basis vectors. Furthermore, from the original Heisenberg dynamics of the quantum fields corresponding functional equations for these state functionals can be derived. Weak mapping is then introduced by a reordering of such generating state functionals according to certain bound state structures. As a result one obtains another state functional which is by definition the image of the weak mapping operator. Since the functional field equations may be viewed as operators acting on the set of generating state functionals, by weak mapping these operators are simultaneously transformed with the states, and this yields new functional field equations which are considered as the algebraic description of the effective dynamics of composite particles in quantum field theory.

So far, weak mapping was confined to the treatment of fully regularized quantum field theories, i.e., singularities of the quantum fields and their subsequent

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renormalization were excluded. In this paper we extend the weak mapping formalism to the case of renormalizable quantum field theories. We first exemplify this treatment for the most simple model of a gauge theory, namely spinor quantum electrodynamics. In forthcoming papers we will proceed to the treatment of nonabelian gauge theories.

In the algebraic formulation of the quantum field dynamics the time evolution of the quantum fields is assumed to be a one-parameter automorphism group of the field algebra. This kind of description of quantum fields is closely related to the Hamilton formalism which is not manifestly invariant, or covariant, respectively. In order to adapt spinor quantum electrodynamics to this formalism, we cannot use a covariant gauge with superfluous field variables. Rather we have to apply a noncovariant gauge with a minimal number of physical field variables, and for spinor quantum electrodynamics the Coulomb gauge seems to be the most appropriate one. Although the Coulomb gauge is a noncovariant gauge, it was recently demonstrated that it is inherently relativistic invariant even in nonabelian theories [7]. We therefore conclude that effective dynamics derived in Coulomb gauge and Hamilton formalism will be inherently relativistic invariant, too, but we will not explicitly prove this property here.

2. Spinor Electrodynamics in Coulomb Gauge

The bare Lagrangian density of spinor electrodynamics reads

$$\mathscr{L} := -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + \tfrac{1}{2}(i\bar{\psi}\gamma^\mu \partial_\mu \psi - i(\partial_\mu \bar{\psi})\gamma^\mu \psi) - m_0 \bar{\psi}\psi + e_0 A_\mu :\bar{\psi}\gamma^\mu\psi: \tag{1}$$

with

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2}$$

and m_0 and e_0 the bare fermion mass and charge. According to Sect. 1 we treat spinor electrodynamics in a non-covariant gauge. With respect to composite particle dynamics under all non-covariant gauges the Coulomb gauge is definitely distinguished, as in this gauge the basic binding force between charged fermions, i.e., the Coulomb force or its potential, respectively, explicitly appears in the Hamiltonian. Furthermore, we can work in this gauge with a minimal number of physical field variables, which enables us to define consistent field quantization rules and to prevent the admixture of unphysical states to physical bound states.

For the algebraic treatment we need the corresponding field equations. If we combine ψ and its charge conjugated field ψ^c into a superspinor field Ψ defined by

$$\Psi_{\alpha\Lambda} := \begin{cases} \psi_\alpha & \text{if } \Lambda = 1 \\ \psi^c_\alpha = C_{\alpha\beta}\bar{\psi}_\beta & \text{if } \Lambda = 2 \end{cases}, \quad C := i\gamma^2\gamma^0, \tag{3}$$

we get from (1)

$$\partial_\nu F^{\mu\nu} + \frac{ie_0}{2} \Psi C \gamma^\mu \sigma^2 \Psi = 0, \tag{4}$$

$$(i\gamma^\mu \partial_\mu - m_0)\Psi + e_0 A_\mu \gamma^\mu \sigma^3 \Psi = 0, \tag{5}$$

where the Pauli matrices σ^μ act on the index Λ.

In addition, the algebraic treatment and the Hamilton formalism need a formulation of the field equations in terms of canonically conjugate variables, in particular with respect to the electromagnetic field. We therefore introduce $E^k := F^{0k}$ into the above field equations. Equations (4) and (5) can then be rewritten as

$$\begin{aligned} i\partial_0 \Psi &= -(i\alpha^k \partial_k - \beta m_0)\Psi - e_0 \gamma^0 \gamma^\mu A_\mu \sigma^3 \Psi, \\ i\partial_0 E^k &= -\tfrac{1}{2} e_0 \Psi C \gamma^k \sigma^2 \Psi + i(\partial_j \partial^k A^j - \partial_j \partial^j A^k), \\ i\partial_0 A^k &= iE^k - i\partial_k A^0, \\ \partial_k E^k &= -i\frac{e_0}{2} \Psi C \gamma^0 \sigma^2 \Psi. \end{aligned} \tag{6}$$

We now impose the gauge condition

$$\partial_k A^k = 0 \tag{7}$$

and decompose E^k and A^k into transversal and longitudinal parts: $E^k = E^k_{tr} + E^k_l$, $A^k = A^k_{tr} + A^k_l$.

The decomposition is performed by the application of projection operators

$$P_{tr} := 1 - \Delta^{-1}\nabla \otimes \nabla\cdot, \quad P_l = \Delta^{-1}\nabla \otimes \nabla\cdot \tag{8}$$

with $P_{tr} V = V_{tr}$ and $P_l V = V_l$. Furthermore it is

$$\nabla \cdot V_{tr} = 0, \quad \nabla \times V_l = 0 \tag{9}$$

for any vector V. In particular, from the gauge condition it follows $A_l = P_l A = 0$, i.e. $A = A_{tr}$.

Applying the projection operators P_{tr} and P_l to (6), we obtain the following set of equations

$$i\partial_0 \Psi = -(i\alpha^k \partial_k - \beta m_0)\Psi - e_0 A^0 \sigma^3 \Psi + e_0 \alpha^k A^k_{tr} \sigma^3 \Psi, \tag{10}$$

$$i\partial_0 E^k_{tr} = -P_{tr} \frac{e_0}{2} \Psi C \gamma^k \sigma^2 \Psi + i\Delta A^k_{tr}, \tag{11}$$

$$i\partial_0 A^k_{\rm tr} = i E^k_{\rm tr}\,, \tag{12}$$

$$i\partial_0 E^k_l = -P_l \frac{e_0}{2}\,\Psi C\gamma^k\sigma^2\Psi\,, \tag{13}$$

$$E^k_l = \partial_k A^0\,, \tag{14}$$

$$\partial_k E^k_l = -i\frac{e_0}{2}\,\Psi C\gamma^0\sigma^2\Psi\,. \tag{15}$$

If we observe current conservation it turns out that (13) depends linearly on (15). Hence in the following we can omit (13). Substitution of (14) into (15) with appropriate boundary conditions yields

$$A^0 = -\frac{i}{2}\,e_0\,\Delta^{-1}\Psi C\gamma^0\sigma^2\Psi\,. \tag{16}$$

By this equation, (14) and (15) are satisfied and we can replace (14) and (15) by (16). Finally we substitute (16) into the spinorial equations (10). Therefore, from (6) the following set of independent equations remains:

$$\begin{aligned} i\partial_0\Psi &= -(i\alpha^k\partial_k - \beta m_0)\Psi \\ &\quad + \frac{i}{2}e_0^2(\Delta^{-1}\Psi C\gamma^0\sigma^2\Psi)\sigma^3\Psi + e_0\alpha^k A^k_{\rm tr}\sigma^3\Psi\,, \\ i\partial_0 E^k_{\rm tr} &= -P_{\rm tr}\frac{e_0}{2}\,\Psi C\gamma^k\sigma^3\Psi + i\Delta A^k_{\rm tr}\,, \qquad (17) \\ i\partial_0 A^k_{\rm tr} &= i E^k_{\rm tr}\,. \end{aligned}$$

In these equations all superfluous field variables are eliminated in favour of a minimal set of independent field variables, namely Ψ, $E^k_{\rm tr}$, $A^k_{\rm tr}$.

With regard to subsequent quantization and renormalization we have to supplement the classical Lagrangian by the corresponding counterterms, and therefore we have to correct the dynamical equation (17), too.

Usually, renormalization is performed by means of multiplicative Z-factors for fields and charge. It can, however, be proven that this multiplicative renormalization can be transformed into a mass and charge renormalization [8]. This gives

$$m_0 = m - \delta m\,; \quad e_0 = e - \delta e\,, \tag{18}$$

where m is the observed mass and e the observed charge of the fermion, and δm and δe are the only counter terms of the theory. As the renormalization can be performed in any gauge, this result must hold in the Coulomb gauge and we can substitute (18) into (17) without any difficulty.

In general, renormalization is properly defined only in perturbation theory. On the other hand we rejected perturbation theory as an inadequate means to treat composite particle dynamics. How this difficulty can be resolved will be seen in the following section. In any case, as a first step we need a perturbation independent algebraic scheme to treat spinor QED.

To perform such an algebraic treatment it is convenient to introduce some abbreviations. First we introduce an additional index η to combine the fields $E^k_{\rm tr}$, $A^k_{\rm tr}$ into a superfield

$$B^k_\eta := \begin{cases} A^k_{\rm tr} & \text{if } \eta = 1 \\ E^k_{\rm tr} & \text{if } \eta = 2 \end{cases}. \tag{19}$$

The vector fields are then characterized by the arguments $\{\mathbf{r}, k, \eta\}$, while the spinor field has the arguments $\{\mathbf{r}, \alpha, \Lambda\}$. For both kinds of arguments we symbolically use only one index $K := \{\mathbf{z}, k, \eta\}$ or $I := \{\mathbf{r}, \alpha, \Lambda\}$, respectively. With these abbreviations (17) can be symbolically rewritten as

$$\begin{aligned} i\partial_0\Psi_I &= D_{II'}\Psi_{I'} + W^K_{II'}B_K\Psi_{I'} + U_I^{I_1I_2I_3}\Psi_{I_1}\Psi_{I_2}\Psi_{I_3}\,, \\ i\partial_0 B_K &= L_{KK'}B_{K'} + J_K^{I_1I_2}\Psi_{I_1}\Psi_{I_2} \end{aligned} \tag{20}$$

with

$$\begin{aligned} D_{II'} &:= -(i\alpha^k\partial_k - \beta m_0)_{\alpha\alpha'}\,\delta(\mathbf{r}-\mathbf{r}')\,\delta_{\Lambda\Lambda'}\,, \\ W^K_{II'} &:= e_0\,\alpha^k_{\alpha\alpha'}\,\delta(\mathbf{r}-\mathbf{r}')\,\delta(\mathbf{r}-\mathbf{z})\,\delta_{1\eta}\,\sigma^3_{\Lambda\Lambda'}\,, \\ U_I^{I_1I_2I_3} &:= -\frac{i}{8\pi}e_0^2\left\{(C\gamma^0)_{\alpha_1\alpha_2}\,\delta_{\alpha\alpha_3}\,\sigma^2_{\Lambda_1\Lambda_2}\,\sigma^3_{\Lambda\Lambda_3} \cdot \frac{\delta(\mathbf{r}_1-\mathbf{r}_2)\,\delta(\mathbf{r}-\mathbf{r}_3)}{|\mathbf{r}-\mathbf{r}_1|}\right\}_{\text{as}(123)}, \qquad (21) \\ L_{KK'} &:= i\,\delta(\mathbf{z}-\mathbf{z}')\,\delta_{kk'}\,\delta_{\eta1}\,\delta_{2\eta'} + i\Delta_z\,\delta(\mathbf{z}-\mathbf{z}')\,\delta_{kk'}\,\delta_{\eta2}\,\delta_{1\eta'}\,, \\ J_K^{I_1I_2} &:= -\tfrac{1}{2}\,e_0\,P^z_{\rm tr}\,\delta(\mathbf{z}-\mathbf{r}_1)\,\delta(\mathbf{z}-\mathbf{r}_2)(C\gamma^k)_{\alpha_1\alpha_2}\,\delta_{2\eta}\,\sigma^2_{\Lambda_1\Lambda_2}\,, \end{aligned}$$

where all symbolic operators are formulated as integral kernels with summation convention, aside from $P^z_{\rm tr}$ which means the action of $P_{\rm tr}$ on the subsequent expression with free index $\mathbf{z}$.

3. Quantization of Spinor Electrodynamics

We turn to the quantization of (20). With respect to the fermion field we impose the canonical equal time anticommutation relation

$$[\Psi_{I_1}, \Psi_{I_2}]^t_+ = A_{I_1I_2}\mathbb{1}\,; \quad A_{I_1I_2} = (C\gamma^0)_{\alpha_1\alpha_2}\,\sigma^1_{\Lambda_1\Lambda_2}\,\delta(\mathbf{r}_1-\mathbf{r}_2)\,. \tag{22}$$

Furthermore we assume for equal times t

$$[B_K, \Psi_I]_-^t = 0\,, \tag{23}$$

$$[B_{K_1}, B_{K_2}]_-^t = C_{K_1K_2}\,\mathbb{1}\,, \tag{24}$$

where $C_{K_1K_2}$ has to be a purely imaginary antisymmetric generalized c-number function, i.e. $C_{K_1K_2} = -C_{K_2K_1}$. Usually, the special form of $C_{K_1K_2}$ is postulated by means of the canonical quantization procedure. However, as will be seen below, $C_{K_1K_2}$ is uniquely determined from the field equations (20) and the fermion anticommutator (22) by the requirement of a consistent quantum theory. Hence, we do not specify the form of the commutator $C_{K_1K_2}$ from the beginning, rather it will come out as a result of our calculation.

In the algebraic formulation of quantum field theory [9] the quantum states $|a\rangle$ of our system can be characterized by the set of matrixelements for equal times t

$$\langle 0|\,\mathscr{A}(\Psi_{I_1}\ldots\Psi_{I_n})_t\,\mathscr{S}(B_{K_1}\ldots B_{K_m})_t\,|a\rangle, \qquad n, m = 1\ldots\infty\,. \tag{25}$$

The operator $\mathscr{A}$ antisymmetrizes the product of field operators $\Psi_{I_1}\ldots\Psi_{I_n}$ with respect to the indices $1,\ldots,n$, whereas the operator $\mathscr{S}$ symmetrizes the boson operators with respect to the indices $1,\ldots,m$. Therefore, we have to establish an equation which allows, at least in principle, the determination of the matrixelements (25). If we assume the existence of a Hamiltonian operator $\mathbf{H}$, we have the following Heisenberg equation:

$$i\partial_t\,\mathscr{A}(\Psi_{I_1}\ldots\Psi_{I_n})_t\,\mathscr{S}(B_{K_1}\ldots B_{K_m})_t = [\mathscr{A}(\Psi_{I_1}\ldots\Psi_{I_n})_t\,\mathscr{S}(B_{K_1}\ldots B_{K_m})_t, \mathbf{H}]_-\,. \tag{26}$$

We require the states $|0\rangle$, $|a\rangle$, which are characterized by a complete set of quantum numbers, to be eigenstates of $\mathbf{H}$. Then we obtain from (26)

$$\Delta E\,\langle 0|\,\mathscr{A}(\Psi_{I_1}\ldots\Psi_{I_n})_t\,\mathscr{S}(B_{K_1}\ldots B_{K_m})_t\,|a\rangle = i\partial_t\,\langle 0|\,\mathscr{A}(\Psi_{I_1}\ldots\Psi_{I_n})_t\,\mathscr{S}(B_{K_1}\ldots B_{K_m})_t\,|a\rangle\,. \tag{27}$$

For brevity we suppress the index t in (25) in the following.

With the help of (20) we can evaluate the right hand side of (27):

$$\begin{aligned} & i\partial_t\,\langle 0|\,\mathscr{A}(\Psi_{I_1}\ldots\Psi_{I_n})_t\,\mathscr{S}(B_{K_1}\ldots B_{K_m})\,|a\rangle \\ &= \sum_{l=1}^{n}\langle 0|\,\mathscr{A}(\Psi_{I_1}\ldots(D_{I_lI'}\Psi_{I'})\ldots\Psi_{I_n})\,\mathscr{S}(B_{K_1}\ldots B_{K_m})\,|a\rangle \\ &+ \sum_{l=1}^{n}\langle 0|\,\mathscr{A}(\Psi_{I_1}\ldots(W_{I_lI'}^{K}\Psi_{I'})\ldots\Psi_{I_n})\,B_K\,\mathscr{S}(B_{K_1}\ldots B_{K_m})\,|a\rangle \\ &+ \sum_{l=1}^{n}\langle 0|\,\mathscr{A}(\Psi_{I_1}\ldots(U_{I_l}^{M_1M_2M_3}\Psi_{M_1}\Psi_{M_2}\Psi_{M_3})\ldots\Psi_{I_n}) \cdot \mathscr{S}(B_{K_1}\ldots B_{K_m})\,|a\rangle \\ &+ \sum_{l=1}^{m}\langle 0|\,\mathscr{A}(\Psi_{I_1}\ldots\Psi_{I_n})\,\mathscr{S}(B_{K_1}\ldots(L_{K_lK_l'}B_{K_l'})\ldots B_{K_m})\,|a\rangle \\ &+ \sum_{l=1}^{m}\langle 0|\,\mathscr{A}(\Psi_{I_1}\ldots\Psi_{I_n})\,\Psi_{M_1}\Psi_{M_2} \cdot \mathscr{S}(B_{K_1}\ldots B_{K_{l-1}}\,J_{K_l}^{M_1M_2}\,B_{K_{l+1}}\ldots B_{K_m})\,|a\rangle \\ &=: T_1 + T_2 + T_3 + T_4 + T_5\,. \end{aligned} \tag{28}$$

In order to get an equation for the matrixelements (25) we have to represent the right hand side of (28) as a linear combination of the matrixelements (25). As a first step in this direction we transform the expressions $T_1,\ldots,T_5$ into a standard representation.

We begin with the evaluation of T_1. By repeated anticommutations we shift $\Psi_{I'}$ to the left, and after relabeling we get

$$T_1 = \sum_{p\in S_n}\frac{\operatorname{sgn}(p)}{n!}\sum_{l=1}^{n} D_{I_{p(1)}I'}\,[\langle 0|\,\Psi_{I'}\Psi_{I_{p(2)}}\ldots\Psi_{I_{p(n)}}\,\mathscr{S}(B_{K_1}\ldots B_{K_m})\,|a\rangle - (l-1)\,A_{I_{p(2)}I'}\,\langle 0|\,\Psi_{I_{p(3)}}\ldots\Psi_{I_{p(n)}}\,\mathscr{S}(B_{K_1}\ldots B_{K_m})\,|a\rangle]\,. \tag{29}$$

With the formula

$$\sum_{\varkappa=1}^{n}(\varkappa-1)\ldots(\varkappa-r) = \frac{1}{r+1}\,n(n-1)\ldots(n-r) \tag{30}$$

we obtain from (29)

$$T_1 = \sum_{p\in S_n}\frac{\operatorname{sgn}(p)}{n!}\,D_{I_{p(1)}I'}\,[n\,\langle 0|\,\Psi_{I'}\Psi_{I_{p(2)}}\ldots\Psi_{I_{p(n)}}\,\mathscr{S}(B_{K_1}\ldots B_{K_n})\,|a\rangle - \tfrac{1}{2}n(n-1)\,A_{I_{p(2)}I'}\,\langle 0|\,\Psi_{I_{p(3)}}\ldots\Psi_{I_{p(n)}}\,\mathscr{S}(B_{K_1}\ldots B_{K_n})\,|a\rangle]\,. \tag{31}$$

The evaluation of T_2 runs along the same lines as the evaluation of T_1. The result is

$$T_2 = \sum_{p \in S_n} \frac{\operatorname{sgn}(p)}{n!} W^K_{I_{p(1)} I'} [n \langle 0 | \Psi_{I'} \Psi_{I_{p(2)}} \dots \Psi_{I_{p(n)}} B_K \mathscr{S}(B_{K_1} \dots B_{K_m}) | a\rangle$$
$$- \tfrac{1}{2} n(n-1) A_{I_{p(2)} I'} \langle 0 | \Psi_{I_{p(3)}} \dots \Psi_{I_{p(n)}} B_K \mathscr{S}(B_{K_1} \dots B_{K_n}) | a\rangle]. \quad (32)$$

We turn to the evaluation of T_3. By repeated anticommutations we shift $\Psi_{M_1}, \Psi_{M_2}, \Psi_{M_3}$ to the left. Then, after relabeling and making use of (30), we have

$$T_3 = \sum_{p \in S_n} \frac{\operatorname{sgn}(p)}{n!} U^{M_1 M_2 M_3}_{I_{p(1)}} [n \langle 0 | \Psi_{M_1} \Psi_{M_2} \Psi_{M_3} \Psi_{I_{p(2)}} \dots \Psi_{I_{p(n)}} \mathscr{S}(B_{K_1} \dots B_{K_n}) | a\rangle$$
$$- \tfrac{3}{2} n(n-1) A_{I_{p(2)} M_1} \langle 0 | \Psi_{M_2} \Psi_{M_3} \Psi_{I_{p(3)}} \dots \Psi_{I_{p(n)}} \mathscr{S}(B_{K_1} \dots B_{K_n}) | a\rangle$$
$$+ n(n-1)(n-2) A_{I_{p(2)} M_1} A_{I_{p(3)} M_3} \langle 0 | \Psi_{M_2} \Psi_{I_{p(4)}} \dots \Psi_{I_{p(n)}} \mathscr{S}(B_{K_1} \dots B_{K_n}) | a\rangle$$
$$+ \tfrac{1}{4} n(n-1)(n-2)(n-3) A_{I_{p(2)} M_1} A_{I_{p(3)} M_2} A_{I_{p(4)} M_3} \langle 0 | \Psi_{I_{p(5)}} \dots \Psi_{I_{p(n)}} \mathscr{S}(B_{K_1} \dots B_{K_n}) | a\rangle]. \quad (33)$$

The evaluation of T_4 is analogous to the evaluation of T_1 and T_2. By repeated commutations we shift $B_{K'}$ to the left. With the help of (30) we obtain after relabeling

$$T_4 = \sum_{p \in S_m} \frac{1}{m!} L_{K_{p(1)} K'} [m \langle 0 | \mathscr{A}(\Psi_{I_1} \dots \Psi_{I_n}) B_{K'} B_{K_{p(2)}} \dots B_{K_{p(m)}} | a\rangle$$
$$+ \tfrac{1}{2} m(m-1) C_{K_{p(2)} K'} \langle 0 | \mathscr{A}(\Psi_{I_1} \dots \Psi_{I_n}) B_{K_{p(3)}} \dots B_{K_{p(m)}} | a\rangle]. \quad (34)$$

We now discuss the term T_5. Anticommuting Ψ_{M_1}, Ψ_{M_2} to the left, relabeling and making use of (30), yields

$$T_5 = \sum_{p \in S_n} \frac{\operatorname{sgn}(p)}{n!} \sum_{\pi \in S_m} \frac{1}{m!} J^{M_1 M_2}_{K_{\pi(1)}} [m \langle 0 | \Psi_{M_1} \Psi_{M_2} \Psi_{I_{p(1)}} \dots \Psi_{I_{p(n)}} B_{K_{\pi(2)}} \dots B_{K_{\pi(m)}} | a\rangle$$
$$+ 2 n m A_{I_{p(1)} M_1} \langle 0 | \Psi_{M_2} \Psi_{I_{p(2)}} \dots \Psi_{I_{p(n)}} B_{K_{\pi(2)}} \dots B_{K_{\pi(m)}} | a\rangle \quad (35)$$
$$- n(n-1) m A_{I_{p(1)} M_1} A_{I_{p(2)} M_2} \langle 0 | \Psi_{I_{p(3)}} \dots \Psi_{I_{p(n)}} B_{K_{\pi(2)}} \dots B_{K_{\pi(m)}} | a\rangle].$$

With (31), (32), (33), (34), and (35) we have brought the right hand side of (28) into standard form, i.e., all operators which are not antisymmetrized or symmetrised, respectively, are shifted to the left.

The next step is to decompose this standard representation into a sum of matrixelements of the form (27). This decomposition can be described in a compact and concise manner by means of functional equations. We introduce the functional sources j_I, b_K and their duals ∂_I, ∂^b_K, which are defined by their anticommutation or commutation relations, respectively,

$$[j_I, j_{I'}]_+ = 0, \quad [j_I, \partial_{I'}]_+ = \delta_{II'},$$
$$[b_K, b_{K'}]_- = 0, \quad [\partial^b_K, b_{K'}]_- = \delta_{KK'}, \quad (36)$$

while all other commutators vanish. For a complete algebraic treatment we have to introduce a functional space. This can be done by defining a Fock space structure with respect to the sources $\{j_I, b_K\}$ with a corresponding source vacuum $|0\rangle_F$:

$$\partial_I |0\rangle_F = \partial^b_K |0\rangle_F = {}_F\langle 0 | j_I = {}_F\langle 0 | b_K = 0. \quad (37)$$

It should be noted that this construction is not in conflict with algebraic representation theory of the original fields. Rather it is only a means for a compact and precise formulation of their algebraic properties.

We substitute (31), (32), (33), (34), and (35) into (28) and (28) into (27), multiply from the right with

$$\frac{i^n\, i^m}{n!\, m!}\, j_{I_1} \dots j_{I_n}\, b_{K_1} \dots b_{K_m}\, |0\rangle_F \tag{38}$$

and perform the summation over n, m from 0 to ∞. Then we get after relabeling

$$\begin{aligned}\Delta E\, |\mathscr{G}(j,b,a)\rangle &= i D_{II'}\, j_I\, |\mathscr{G}_{I'}(j,b,a)\rangle + \tfrac{1}{2} D_{I_1 I}\, A_{I_2 I}\, j_{I_1} j_{I_2}\, |\mathscr{G}(j,b,a)\rangle + i W^K_{I_1 I}\, j_{I_1}\, |\mathscr{G}^K_I(j,b,a)\rangle \\ &+ \tfrac{1}{2} W^K_{I_1 I}\, A_{I_2 I}\, j_{I_1} j_{I_2}\, |\mathscr{G}^K(j,b,a)\rangle + i U_I^{M_1 M_2 M_3}\, j_I \Big[|\mathscr{G}_{M_1 M_2 M_3}(j,b,a)\rangle - \frac{3j}{2} A_{I' M_1}\, j_{I'}\, |\mathscr{G}_{M_2, M_3}(j,b,a)\rangle \\ &- A_{I_1 M_1}\, A_{I_2 M_3}\, j_{I_1} j_{I_2}\, |\mathscr{G}_{M_2}(j,b,a)\rangle - \frac{i}{4} A_{I_1 M_1}\, A_{I_2 M_2}\, A_{I_3 M_3}\, j_{I_1} j_{I_2} j_{I_3}\, |\mathscr{G}(j,b,a)\rangle \Big] \\ &+ i b_{K'}\, L_{K'K}\, |\mathscr{G}^K(j,b,a)\rangle - \tfrac{1}{2} L_{K_1 K}\, C_{K_2 K}\, b_{K_1} b_{K_2}\, |\mathscr{G}(j,b,a)\rangle \\ &+ i J_K^{M_1 M_2}\, b_K \Big[|\mathscr{G}_{M_1, M_2}(j,b,a)\rangle + 2i A_{I M_1}\, j_I\, |\mathscr{G}_{M_2}(j,b,a)\rangle + A_{I M_1}\, A_{I' M_2}\, j_I\, j_{I'}\, |\mathscr{G}(j,b,a)\rangle \Big],\end{aligned} \tag{39}$$

where we use the definition

$$|\mathscr{G}^{K_1 \dots K_s}_{M_1 \dots M_r}(j,b,a)\rangle := \sum_{n,m=0}^{\infty} \frac{i^n\, i^m}{n!\, m!} \langle 0|\, \Psi_{M_1} \dots \Psi_{M_r}\, \Psi_{I_1} \dots \Psi_{I_n}\, B_{K_1} \dots B_{K_s}\, B_{K'_1} \dots B_{K'_m}\, |a\rangle\, j_{I_1} \dots j_{I_n}\, b_{K'_1} \dots b_{K'_m}\, |0\rangle_F\,. \tag{40}$$

For the further evaluation we prove the following theorem:

Theorem 1. *The recursion formula*

$$\begin{aligned}\langle 0|\, \Psi_M\, \mathscr{A}(\Psi_{I_1} \dots \Psi_{I_n})\, \mathscr{S}(B_{K_1} \dots B_{K_m})\, |a\rangle &= \sum_{p \in S_n} \frac{\operatorname{sgn}(p)}{n!} \frac{1}{n+1} \Bigg[\sum_{\varkappa=1}^{n+1} (-1)^{\varkappa+1} \langle 0|\, \Psi_{I_{p(1)}} \dots \Psi_M\, \Psi_{I_{p(\varkappa)}} \dots \Psi_{I_{p(n)}} \\ &\times \mathscr{S}(B_{K_1} \dots B_{K_m})\, |a\rangle + \sum_{\varkappa=1}^{n+1} \varkappa\, A_{M I_{p(1)}} \langle 0|\, \Psi_{I_{p(2)}} \dots \Psi_{I_{p(n)}}\, \mathscr{S}(B_{K_1} \dots B_{K_m})\, |a\rangle \Bigg],\end{aligned} \tag{41}$$

which generates the relation between non-antisymmetric standard products and antisymmetric products of field operators can be expressed by the functional relation

$$|\mathscr{G}_M(j,b,a)\rangle = \left(\frac{1}{j}\, \partial_M + \frac{i}{2} A_{MM'}\, j_{M'} \right) |\mathscr{G}(j,b,a)\rangle\,. \tag{42}$$

Proof: We use the identity

$$\begin{aligned}&\sum_{p \in S_n} \frac{\operatorname{sgn}(p)}{n!} \langle 0|\, \Psi_M\, \Psi_{I_{p(1)}} \dots \Psi_{I_{p(n)}}\, \mathscr{S}(B_{K_1} \dots B_{K_m})\, |a\rangle \\ &\qquad = \sum_{p \in S_n} \frac{\operatorname{sgn}(p)}{n!} \frac{1}{n+1} \sum_{l=1}^{n+1} \langle 0|\, \Psi_M\, \Psi_{I_{p(1)}} \dots \Psi_{I_{p(n)}}\, \mathscr{S}(B_{K_1} \dots B_{K_m})\, |a\rangle\end{aligned} \tag{43}$$

and shift in the summand $l=\varkappa$ on the right hand side of (43) the operator Ψ_M by repeated anticommutations in the position $\varkappa$, i.e. to

$$\langle 0|\, \Psi_{I_{p(1)}} \dots \Psi_{I_{p(\varkappa-1)}}\, \Psi_M\, \Psi_{I_{p(\varkappa)}} \dots \Psi_{I_{p(n)}}\, \mathscr{S}(B_{K_1} \dots B_{K_m})\, |a\rangle\,.$$

After relabeling we get (41).

If we multiply (41) from the right with

$$\frac{i^n\, i^m}{n!\, m!}\, j_{I_1} \dots j_{I_n}\, b_{K_1} \dots b_{K_m} |0\rangle_F \,,$$

we obtain, with the use of (30) and after relabeling, (42). ◇

The relation between non-symmetric standard products and symmetric products of boson field operators is given by

Theorem 2.

$$|\mathscr{G}^K(j, b, a)\rangle = \left(\frac{1}{i}\,\partial^b_K + \frac{i}{2}\, C_{KK'}\, b_{K'}\right) |\mathscr{G}(j, b, a)\rangle \,. \tag{44}$$

Proof: The proof is analogous to the proof of Theorem 1 and will be omitted for brevity. ◇

Theorem 3. *For the general state* (40) *the relation*

$$|\mathscr{G}^{K_1 \dots K_s}_{M_1 \dots M_r}(j, b, a)\rangle = \left(\frac{1}{i}\,\partial_{M_r} + \frac{i}{2}\, A_{M_r M'_r}\, j_{M'_r}\right) \dots \left(\frac{1}{i}\,\partial_{M_1} + \frac{i}{2}\, A_{M_1 M'_1}\, j_{M'_1}\right)$$
$$\cdot \left(\frac{1}{i}\,\partial^b_{K_s} + \frac{i}{2}\, C_{K_s K'_s}\, b_{K'_s}\right) \dots \left(\frac{1}{i}\,\partial^b_{K_1} + \frac{i}{2}\, C_{K_1 K'_1}\, b_{K'_1}\right) |\mathscr{G}(j, b, a)\rangle \tag{45}$$

holds.

Proof: Repeated application of Theorem 1 and Theorem 2. ◇

With the help of Theorem 3 we can rearrange (39) and obtain

$$\Delta E\, |\mathscr{G}(j, b, a)\rangle = \Big[j_I D_{II'}\, \partial_{I'} - i\, j_I W^K_{II'}\, \partial_{I'}\, \partial^b_K + \frac{i}{2}\, j_I\, W^K_{II'}\, C_{KK'}\, b_{K'}\, \partial_{I'}$$
$$+ j_I U_I^{M_1 M_2 M_3} (\partial_{M_1}\, \partial_{M_2}\, \partial_{M_3} - \tfrac{1}{4}\, A_{M_2 I_2}\, A_{M_1 I_1}\, j_{I_2} j_{I_1}\, \partial_{M_3}) \tag{46}$$
$$+ b_K L_{KK'}\, \partial^b_{K'} + i J_K^{M_1 M_2}\, b_K (\partial_{M_1}\, \partial_{M_2} + A_{I_2 M_1}\, j_{I_2}\, \partial_{M_2} + \tfrac{1}{4}\, A_{I M_1}\, A_{I' M_2}\, j_I j_{I'}) \Big] |\mathscr{G}(j, b, a)\rangle \,.$$

In order to derive the above functional equation, we began our calculation with the matrix elements (25). But we can also start our calculation with matrixelements where the boson operators and the fermion operators are arrayed in reverse order, i.e. with matrixelements of the form

$$\langle 0|\, \mathscr{S}(B_{K_1} \dots B_{K_m})\, \mathscr{A}(\Psi_{I_1} \dots \Psi_{I_n})\, |a\rangle \,. \tag{47}$$

Then by the same reasoning which leads to (46) we get

$$\Delta E\, |\hat{\mathscr{G}}(j, b, a)\rangle = \Big[j_I D_{II'}\, \partial_{I'} - i\, j_I W^K_{II'}\, \partial_{I'}\, \partial^b_K - \frac{i}{2}\, j_I\, W^K_{II'}\, C_{KK'}\, b_{K'}\, \partial_{I'}$$
$$+ j_I U_I^{M_1 M_2 M_3} (\partial_{M_1}\, \partial_{M_2}\, \partial_{M_3} - \tfrac{1}{4}\, A_{M_2 I_2}\, A_{M_1 I_1}\, j_{I_2} j_{I_1}\, \partial_{M_3}) \tag{48}$$
$$+ b_K L_{KK'}\, \partial^b_{K'} + i J_K^{M_1 M_2}\, b_K (\partial_{M_1}\, \partial_{M_2} - A_{I_2 M_1}\, j_{I_2}\, \partial_{M_2} + \tfrac{1}{4}\, A_{I M_1}\, A_{I' M_2}\, j_I j_{I'}) \Big] |\hat{\mathscr{G}}(j, b, a)\rangle \,.$$

with

$$|\hat{\mathscr{G}}(j, b, a)\rangle := \sum_{n, m=0}^{\infty} \frac{i^n\, i^m}{n!\, m!} \langle 0|\, \mathscr{S}(B_{K_1} \dots B_{K_m})\, \mathscr{A}\{\Psi_{I_1} \dots \Psi_{I_n}\}\, |a\rangle\, j_{I_1} \dots j_{I_n}\, b_{K_1} \dots b_{K_m} |0\rangle_F \,. \tag{49}$$

According to (23), the operator Ψ_I commutes with B_K, and therefore we have

$$|\hat{\mathscr{G}}(j,b,a)\rangle = |\mathscr{G}(j,b,a)\rangle\,. \tag{50}$$

From (50) we infer that the left hand sides of (46) and (48) must be identical. For consistency the right hand sides must also be identical which, by comparison, leads to the consistency condition

$$C_{KK'}\, W^{K'}_{I_1 I_2} = 2\, A_{I_1 I}\, J^{II_2}_{K}\,. \tag{51}$$

Equation (51) relates the anticommutator $A_{II'}$ and the dynamics with the commutator $C_{KK'}$ of the boson operators. If we substitute (51) into (46) or (48), respectively, we finally obtain

$$\begin{aligned}\Delta E\,|\mathscr{G}(j,b,a)\rangle = {} & [j_I D_{II'}\,\partial_{I'} - i\, j_I\, W^K_{II'}\,\partial_{I'}\,\partial^b_K + b_K\, L_{KK'}\,\partial^b_{K'} \\ & + j_I\, U_I^{M_1 M_2 M_3}(\partial_{M_1}\partial_{M_2}\partial_{M_3} - \tfrac{1}{4} A_{M_2 I_2} A_{M_1 I_1}\, j_{I_2}\, j_{I_1}\,\partial_{M_3}) \\ & + i\, J_K^{M_1 M_2}\, b_K(\partial_{M_1}\partial_{M_2} + \tfrac{1}{4} A_{IM_1} A_{I'M_2}\, j_I\, j_{I'})]\,|\mathscr{G}(j,b,a)\rangle\,.\end{aligned} \tag{52}$$

By projection with

$$\frac{1}{i^m\, i^n}\;{}_F\langle 0|\,\partial^b_{K_m}\ldots\partial^b_{K_1}\,\partial_{I_n}\ldots\partial_{I_1}$$

we immediately obtain from (52) the equations for the matrix elements (25).

To start calculations which include renormalization we have to substitute (18) into (52). Equation (52) is valid beyond perturbation theory and is thus suitable for the discussion of general properties of spinor quantum electrodynamics.

Theorem 4. *The consistency equation* (51), *together with the field dynamics* (17), *uniquely determines the commutator* $C_{KK'}$.

Proof: If we substitute the expressions for $A_{II'}$, $W^K_{II'}$, and $J^K_{II'}$ into (51), we get

$$\begin{aligned}&\int d^3z'\, C\begin{pmatrix} z & z' \\ k & k' \\ \eta & \eta' \end{pmatrix} e_0\, \alpha^{k'}_{\alpha_1\alpha_2}\,\delta(\boldsymbol{r}_1-\boldsymbol{r}_2)\,\delta(\boldsymbol{r}_1-\boldsymbol{z}')\,\sigma^3_{\Lambda_1\Lambda_2}\,\delta_{1\eta'} \\ &= -2\int d^3r\,(C\gamma^0)_{\alpha_1\alpha}\,\sigma^1_{\Lambda_1\Lambda}\,\delta(\boldsymbol{r}_1-\boldsymbol{r})\,\frac{e_0}{2}\,P^{z}_{\rm tr}\,\delta(\boldsymbol{z}-\boldsymbol{r})\,\delta(\boldsymbol{z}-\boldsymbol{r}_2)\,(C\gamma^k)_{\alpha\alpha_2}\,\sigma^2_{\Lambda\Lambda_2}\,\delta_{2\eta}\,.\end{aligned} \tag{53}$$

After performing the summations and integrations, we obtain

$$C\begin{pmatrix} z & r_1 \\ k & k' \\ \eta & 1 \end{pmatrix} = -\,i\,P^{z}_{\rm tr}\,\delta(\boldsymbol{z}-\boldsymbol{r}_1)\,\delta_{kk'}\,\delta_{2\eta}\,. \tag{54}$$

If we choose $\eta=1$, we infer from (54)

$$[A^k_{\rm tr}(\boldsymbol{z}), A^{k'}_{\rm tr}(\boldsymbol{r}_1)]_- = 0\,, \tag{55}$$

and if we choose $\eta=2$, we get

$$[E^k_{\rm tr}(\boldsymbol{z}), A^{k'}_{\rm tr}(\boldsymbol{r}_1)]_- = -\,i\,P^{z}_{\rm tr}\,\delta(\boldsymbol{z}-\boldsymbol{r}_1)\,\delta_{kk'}\,. \tag{56}$$

The commutator relation between the transversal parts of the electric field cannot be read off directly from (51). However, we can deduce it with the help of (55), (56) and the dynamics. From (56) we obtain

$$\begin{aligned}&\partial_0[E^k_{\rm tr}(\boldsymbol{z}), A^{k'}_{\rm tr}(\boldsymbol{r}_1)]_- \\ &= [\partial_0 E^k_{\rm tr}(\boldsymbol{z}), A^{k'}_{\rm tr}(\boldsymbol{r}_1)]_- + [E^k_{\rm tr}(\boldsymbol{z}), \partial_0 A^{k'}_{\rm tr}(\boldsymbol{r}_1)]_- = 0\,.\end{aligned} \tag{57}$$

Substitution of the field equations (17) into (57) gives

$$\left[\frac{ie_0}{2}\,P^{z}_{\rm tr}\,\Psi(\boldsymbol{z})\,\gamma^k\sigma^2\,\Psi(\boldsymbol{z}) + \Delta_z A^k_{\rm tr}(\boldsymbol{z}), A^j_{\rm tr}(\boldsymbol{r})\right]_- + [E^k_{\rm tr}(\boldsymbol{z}), E^j_{\rm tr}(\boldsymbol{r})]_- = 0\,. \tag{58}$$

Hence, with (58), (23) and (55) we obtain

$$[E^k_{\rm tr}(\boldsymbol{z}), E^j_{\rm tr}(\boldsymbol{r})]_- = 0\,. \tag{59}$$

Equations (55), (56) and (59) are the well known canonical commutation relations for the boson operators in Coulomb gauge. We have derived these relations from the requirement

$$\partial_0(\Psi_I B_K) = \partial_0(B_K \Psi_I)\,, \tag{60}$$

which, together with (22), (23) and the dynamics lead to (51). ◇

4. Composite Particle Dynamics

In spinor QED the fermions are commonly identified with electrons and positrons. Due to decay channels into gamma quanta no stable boundstates between electrons and positrons exist, while boundstates between electrons do not occur due to Coulomb repulsion. On the other hand, boundstates for two kinds of fermions, for instance baryons and leptons do exist and can be treated along these lines. For simplicity we consider the electron positron case. According to the general scheme of the composite particle dynamics we introduce boundstates by considering the eigenstates of the fermionic diagonalpart of the functional energy equation. As will be demonstrated, for this part the gamma decay channels are closed and there appear truly boundstates between electrons and positrons. If afterwards with these states the weak mapping procedure is performed, this stability is destroyed by interaction terms which appear as a consequence of this weak mapping. Thus the stability is an intermediate step which is afterwards removed by the theory. Therefore, in principle the weak mapping theorems can also be used for the description of decay processes.

Boundstates in terms of generating functionals are described by corresponding matrix elements. If generating functionals are decomposed by the rules of matrix mechanics or algebraic theory, respectively, one obtains correlated and uncorrelated matrix elements. While the former describe genuine fermion processes, the latter represent contributions of the vacuum to the field dynamics. In order to have a "pure" composite particle theory such vacuum contributions have to be removed from the beginning. In particular for the treatment of two-fermion boundstates the uncorrelated two-fermion propagators have to be eliminated from the generating functional. The propagators which appear in the generating functional are referred to the physical vacuum, i.e., they are the renormalized propagators. But the renormalized propagator which corresponds to the exact solution of the vacuum state is just (by renormalization definition) the ordinary *free* electron-positron propagator with e and m. Thus we apply the transformation

$$|\mathscr{G}(j, b, a)\rangle = \exp(-\tfrac{1}{2} j_{I_1} F_{I_1 I_2} j_{I_2}) \, |\mathscr{F}(j, b, a)\rangle . \quad (61)$$

where $F_{I_1 I_2}$ is the fermion propagator for equal times. Equation (61) induces the replacement of ∂_I in (52) by

$$d_I := \partial_I - F_{I I_1} j_{I_1} . \quad (62)$$

The transformed equation (52) reads

$$\begin{aligned}
\Delta E \, |\mathscr{F}(j, b, a)\rangle
&= \{ D_{I_1 I_2} j_{I_1} \partial_{I_2} - Z_{I_1 I} F_{I I_2} j_{I_1} j_{I_2} - i W^K_{I I'} j_I \partial_{I'} \partial^b_K \\
&\quad + i W^K_{I_1 I} F_{I I_2} j_{I_1} j_{I_2} \partial^b_K + L_{K K'} b_K \partial^b_{K'} \\
&\quad + U^{I_2 I_3 I_4}_{I_1} [j_{I_1} \partial_{I_4} \partial_{I_3} \partial_{I_2} - 3 F_{I_4 I} j_{I_1} j_I \partial_{I_3} \partial_{I_2} \\
&\quad + (\tfrac{1}{4} A_{I_4 I} A_{I_3 I'} + 3 F_{I_4 I} F_{I_3 I'}) j_{I_1} j_I j_{I'} \partial_{I_2} \qquad (63) \\
&\quad - (\tfrac{1}{4} A_{I_4 I} A_{I_3 I'} F_{I_2 I''} + F_{I_4 I} F_{I_3 I'} F_{I_2 I''}) j_{I_1} j_I j_{I'} j_{I''}] \\
&\quad + i J^{M_1 M_2}_K b_K [\partial_{M_1} \partial_{M_2} - 2 F_{M_1 I} j_I \partial_{M_2} \\
&\quad + \tfrac{1}{4} A_{I_1 M_1} A_{I_2 M_2} j_{I_1} j_{I_2}] \} \, |\mathscr{F}(j, b, a)\rangle ,
\end{aligned}$$

where

$$Z_{I_1 I_2} := - \delta m \gamma^0_{\alpha_1 \alpha_2} \delta_{A_1 A_2} \delta(\mathbf{r}_1 - \mathbf{r}_2) \quad (64)$$

and

$$|\mathscr{F}(j, b, a)\rangle =: \sum_{n, m = 0}^{\infty} \frac{i^n i^m}{m! \, n!} \varphi(I_1 \dots I_n, K_1 \dots K_m | a) \cdot j_{I_1} \dots j_{I_n} b_{K_1} \dots b_{K_m} |0\rangle_F . \quad (65)$$

According to weak mapping theory we define bound states to be solutions of the normal-transformed equation (63) with respect to their fermionic diagonal part. This diagonal part equation reads

$$\Delta E' \, |\mathscr{F}\rangle^d = (D_{I I'} j_I \partial_{I'} - 3 U^{I_2 I_3 I_4}_{I_1} F_{I_4 I} j_{I_1} j_I \partial_{I_3} \partial_{I_2}) |\mathscr{F}\rangle^d , \quad (66)$$

where m_0 and e_0 are replaced by (18) and all counterterm expressions with δm and δe are excluded from (66).

The set of solutions (66) can be assumed to be a complete set in $\mathbb{R}^{3n}$ ($n = 1, \dots$), i.e., they describe all possible n-fermion states. In particular we denote the set of two-particle solutions by

$$\{ |\mathscr{F}_k\rangle^d = C^{I_1 I_2}_k j_{I_1} j_{I_2} |0\rangle_F \} . \quad (67)$$

This set contains bound states as well as scattering states. In spite of the completeness, it is, however, in general *not* an orthogonal set since the eigenvalue equation (66) is *not* a Schrödinger equation and the solutions $\{C^{I_1 I_2}_k\}$ in general have *not* the meaning of quantum mechanical probability amplitudes. The non-Schrödinger properties of the set $\{C^{I_1 I_2}_k\}$ are of no relevance for the weak mapping theorem. It suffices to introduce a dual set $\{R^k_{I_1 I_2}\}$ which is defined by the orthogonality and completeness relations

$$\tfrac{1}{2} \sum_{I_1 I_2} R^{k_1}_{I_1 I_2} C^{I_1 I_2}_{k_2} = \delta^{k_1}_{k_2}, \qquad (68)$$

$$\sum_k R^k_{I_1 I_2} C^{I'_1 I'_2}_k = (\delta^{I'_1}_{I_1} \delta^{I'_2}_{I_2} - \delta^{I'_2}_{I_1} \delta^{I'_1}_{I_2}).$$

As the initial set $\{C^{I_1 I_2}_k\}$ is explicitly known, the dual set $\{R^k_{I_1 I_2}\}$ can explicitly be constructed.

We now study the functional mapping of (63) for the simplest case of a transformation to a two-fermion bound state theory. In order to perform such a mapping we have to define a bound state functional

$$|\mathscr{B}(b,c,a)\rangle := \sum_{h,n} \frac{1}{h!\,n!} \varrho(K_1 \dots K_h, k_1 \dots k_n | a) \cdot b_{K_1} \dots b_{K_h} c_{k_1} \dots c_{k_n} |0\rangle_F, \qquad (69)$$

where the c_k are independent bosonic sources which carry the quantum numbers of the two-particle solutions $C^{I_1 I_2}_k$, and the relation between (65) and (69) is given by

$$\varphi(K_1 \dots K_h, I_1 \dots I_{2n} | a) := \frac{(-1)^n (2n)!}{4^n\, n!} C^{\{I_1 I_2}_{k_1} \dots C^{I_{2n-1} I_{2n}\}}_{k_n} \cdot \varrho(K_1 \dots K_h, k_1 \dots k_n | a), \qquad (70)$$

where $\{\ \} \equiv (-1)^P/(2n)!$ permutations. If the set $\{C^{II'}_k\}$ is a complete set, then there is a one-to-one correspondence between the φ- and the ϱ-coefficients of (65) and (69). However, in the case of a pure bound state dynamics we consider by definition only those states $|a\rangle$ for which the expansion (70) contains only these bound states and nothing else, i.e. no two-fermion scattering states which also appear in the complete set $\{C^{II'}_k\}$.

By (70) a mapping between the elementary QED state functionals (65) and QED fermionic bound state functionals (69) is established. The consequences of such a mapping with respect to the functional equation (63) were formulated by a weak mapping theorem [6]. This theorem was proven for a pure spinor theory, but the proof can be taken over to QED without any modifications as far as fermionic bound states are considered.

In its full length the weak mapping theorem contains all exchange effects of the bound states which are due to their fermionic substructure. In order to reduce these rather lengthy expressions, in [10] a short-cut calculation was given which leads to a better manageable version of the theory. It was demonstrated in [6] that this short-cut calculation can be justified if the bound state wave functions are highly concentrated. However, the corresponding estimates are rather complicated and should be completed by other criteria. For instance, one should expect that for low densities of the bound states, i.e. in ideal gas approximation for the composite particles, exchange effects can be neglected. Although this criterion has not yet been investigated, we assume that in quantum electrodynamics of bound states there exist experimental situations where it can be realized. Hence, for a first draft of composite particle theory in quantum electrodynamics, for brevity we ignore exchange effects in the following and apply the short-cut calculation technique of [10], which was the first approach to weak mapping.

According to [10] we have to make the identification

$$|\mathscr{B}(b,c,a)\rangle = |\mathscr{F}(j,b,a)\rangle, \qquad (71)$$

which implies

$$c_k = C^{II'}_k j_I j_{I'}, \qquad (72)$$

i.e., in the short-cut approach the sources j_I, c_k are not considered to be independent. With (71) and (72) we can work out the functional derivative

$$\partial_I |\mathscr{F}(j,b,a)\rangle \equiv \partial_I |\mathscr{B}(b,c,a)\rangle, \qquad (73)$$

which yields by direct calculation the functional chain rule

$$\partial_{I_1} |\mathscr{B}(b,c,a)\rangle = C^{I_1 I}_k j_I \frac{\delta}{\delta c_k} |\mathscr{B}(b,c,a)\rangle, \qquad (74)$$

where $\delta/\delta c_k$ is the dual of c_k.

Successive application of (74) yields

$$\partial_{I_2} \partial_{I_1} |\mathscr{B}(b,c,a)\rangle = -C^{I_1 I}_k C^{I_2 I'}_{k'} j_I j_{I'} \frac{\delta}{\delta c_k} \frac{\delta}{\delta c_{k'}} |\mathscr{B}(b,c,a)\rangle + C^{I_1 I_2}_k \frac{\delta}{\delta c_k} |\mathscr{B}(b,c,a)\rangle, \qquad (75)$$

$$\partial_{I_3} \partial_{I_2} \partial_{I_1} |\mathscr{B}(b,c,a)\rangle \qquad (76)$$

$$= -C^{I_1 I}_k C^{I_2 I'}_{k'} C^{I_3 I''}_{k''} j_I j_{I'} j_{I''} \frac{\delta}{\delta c_k} \frac{\delta}{\delta c_{k'}} \frac{\delta}{\delta c_{k''}} |\mathscr{B}(b,c,a)\rangle - 3 C^{I_1 I_3}_k C^{I_2 I'}_{k'} j_{I'} \frac{\delta}{\delta c_k} \frac{\delta}{\delta c_{k'}} |\mathscr{B}(b,c,a)\rangle.$$

With (71), (74), (75), (76) and the relations

$$j_{I_1} j_{I_2} = R^k_{I_1 I_2} c_k \,, \quad j_{I_1} j_{I_2} j_{I_3} j_{I_4} = R^{k_1}_{\{I_1 I_2} R^{k_2}_{I_3 I_4\}} c_{k_1} c_{k_2} \tag{77}$$

we can map the functional equation (63) for $|\mathscr{F}(j, b, a)\rangle$ onto a functional equation for the functional state $|\mathscr{B}(b, c, a)\rangle$. The result is

$$\begin{aligned}
\Delta E\,|\mathscr{B}(b,c,a)\rangle = \Big\{ & R^{k'}_{II''} D_{II'} C^{I'I''}_k c_{k'} \frac{\delta}{\delta c_k} - R^k_{II''} Z_{II'} F_{I'I''} c_k - i R^k_{II''} W^K_{II'} C^{I'I''}_{k'} c_k \frac{\delta}{\delta c_{k'}} \partial^b_K \\
& + i R^k_{II''} W^K_{II'} F_{I'I''} c_k \partial^b_K + b_K L_{KK'} \partial^b_K + U^{I_2 I_3 I_4}_{I_1} \Big[R^{k_1}_{\{I_1 I} R^{k_2}_{I'I''\}_{\mathrm{as}}} C^{I_2 I}_k C^{I_3 I'}_{k'} C^{I_4 I''}_{k''} c_{k_1} c_{k_2} \frac{\delta}{\delta c_k} \frac{\delta}{\delta c_{k'}} \frac{\delta}{\delta c_{k''}} \\
& - 3 R^{k_1}_{I_1 I'} C^{I_2 I_4}_k C^{I_3 I'}_{k'} c_{k_1} \frac{\delta}{\delta c_k} \frac{\delta}{\delta c_{k'}} + 3 R^{k_1}_{\{I_1 I} R^{k_2}_{I'I''\}_{\mathrm{as}}} F_{I_4 I} C^{I_2 I'}_k C^{I_3 I''}_{k'} c_{k_1} c_{k_2} \frac{\delta}{\delta c_k} \frac{\delta}{\delta c_{k'}} - 3 R^{k_1}_{I_1 I} F_{I_4 I} C^{I_2 I_3}_k c_{k_1} \frac{\delta}{\delta c_k} \\
& + \left(3 F_{I_4 I} F_{I_3 I'} + \tfrac{1}{4} A_{I_4 I} A_{I_3 I'}\right) R^{k_1}_{\{I_1 I} R^{k_2}_{I'I''\}_{\mathrm{as}}} C^{I_2 I''}_k c_{k_1} c_{k_2} \frac{\delta}{\delta c_k} - \left(F_{I_4 I} F_{I_3 I'} + \tfrac{1}{4} A_{I_4 I} A_{I_3 I'}\right) F_{I_2 I''} R^{k_1}_{\{I_1 I} R^{k_2}_{I'I''\}_{\mathrm{as}}} c_{k_1} c_{k_2} \Big] \\
& + i J^{II'}_K b_K \Big[\tfrac{1}{4} A_{I_1 I} A_{I_2 I'} R^k_{I_1 I_2} c_k - 2 R^k_{I_1 I_2} F_{II_1} C^{I'I_2}_{k'} c_k \frac{\delta}{\delta c_{k'}} \\
& - R^{k_1}_{I_1 I_2} C^{I'I_1}_k C^{II_2}_{k'} c_{k_1} \frac{\delta}{\delta c_k} \frac{\delta}{\delta c_{k'}} + C^{I'I}_k \frac{\delta}{\delta c_k} \Big] \Big\} |\mathscr{B}(b,c,a)\rangle \,.
\end{aligned} \tag{78}$$

Apart from exchange effects, (78) is strictly valid without any reference to perturbation theory.

In order to properly renormalize the bound state theory we bring perturbation theory into play by treating all bound state–bound state interactions nonperturbatively, but use perturbation theory for the interaction of the transversal electromagnetic field with the bound states. In this way the contradiction between nonperturbative functional equations and perturbative renormalization is removed. It remains to give a comment with respect to renormalization for bound states:

This question is concerned with the determination of the values of the counterterms δm and δe if we apply the weak mapping procedure, i.e. transform quantum electrodynamics into a bound state dynamics. Primarily, counterterms arise from the renormalization of free electron (positron) mass and charge. Additionally, they are destined to remove all divergencies from the theory under consideration. As in the renormalized equations the empirical mass m and charge e are fixed, it follows that the counterterms are uniquely determined by the condition that the calculation of the selfenergy and electromagnetic coupling constant of the dressed electron (and positron) may not alter the original empirical values of m and e; i.e. δm and δe have to compensate all expressions which would change these empirical values. It is part of the renormalization theory of quantum electrodynamics that simultaneously by this condition also all infinities from the interaction representation are removed.

Now we consider weak mapping. For instance, we assume a mapping by the two-fermion states

$$c_k := \int C_k \begin{pmatrix} \boldsymbol{r} & \boldsymbol{r}' \\ Z_1 & Z_2 \end{pmatrix} j_{Z_1}(\boldsymbol{r})\, j_{Z_2}(\boldsymbol{r}')\, \mathrm{d}^3 r\, \mathrm{d}^3 r' \,, \tag{79}$$

where $\left\{ C_k \begin{pmatrix} \boldsymbol{r} & \boldsymbol{r}' \\ Z_1 & Z_2 \end{pmatrix} k = \text{quantum number} \right\}$ is a complete set of two-fermion wave functions. Among this set we have (Coulomb problem!) bound state and scattering state solutions. The scattering state solutions have ingoing or outgoing free electrons (positrons). By dressing these particles with vacuum- and radiative-corrections we have the same situation as in the case of the interaction representation: The counterterms have to compensate all terms which would alter the already used empirical values m and e. In this way we see that also in the bound state representation the counterterms δm and δe are already fixed by the interaction representation. This means that there is no freedom with respect to bound states for a renewed adaption of the counterterms, as bound states and scattering states occur in one map. On the other hand it must be guaranteed that by this choice for δm and δe also the radiative correction singularities for bound states are compensated. This is *not* selfevident. However, if this should not be the case, there is no escape of the conclusion that renormalization cannot be applied to bound state problems.

In the literature it is demonstrated for hydrogen bound states that this procedure indeed works [11], and for one-electron bound states the Furry picture [12] is used for renormalization. A general proof of renormalizability for bound states, however, seems not to be available.

Let us give an argument that renormalizability may also work in the general case. The argument depends crucially on the possibility of a Hamiltonian, i.e. a Schrödinger representation of quantum electrodynamics. For this representation it suffices to consider all states in a one-time representation which is used as the basis for the weak mapping procedure. In this representation for instance (79) can be resolved into a Fourier series

$$C_k\begin{pmatrix} \boldsymbol{r} & \boldsymbol{r}' \\ Z_1 & Z_2 \end{pmatrix} = \int \tilde{C}_k\begin{pmatrix} \boldsymbol{p} & \boldsymbol{p}' \\ Z_1 & Z_2 \end{pmatrix} e^{-i\boldsymbol{p}\boldsymbol{r} - i\boldsymbol{p}'\boldsymbol{r}'} \, \mathrm{d}^3p \, \mathrm{d}^3p', \tag{80}$$

i.e. a plane wave representation which is simultaneously the basis of the pure interaction representation. Thus, in the Hamiltonian representation the bound state representation and the interaction representation are referred to the same basis system. Thus, if renormalization works in this basis system it must necessarily work in both representations.

Finally we want to remark that the attempt to introduce dressed electrons and positrons by weak mapping already from the beginning, and only afterwards study the formation of bound states will not solve the above mentioned renormalization problem but is much more difficult to perform. In this way the weak mapping with hard-core states seems to be the most promising approach to the discussion of bound state dynamics in renormalizable spinor quantum electrodynamics.

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